It is also concluded in Ref. 1 that the attitude determination accuracy will degrade with time, due to the instability of the combined estimator-regulator system. This conclusion is not correct, and is contradicted by comments following Eq. (9). Though elements of the state covariance matrix (and the state estimate covariance matrix) will grow without bound as time becomes large, the elements of the state estimate error covariance matrix will remain finite. This is guaranteed by the fact that the two-state-variable system is observable with measurements of ψ and the state is controllable by the process noise.[3]

It is finally concluded in Ref. 1 that this instability of the combined estimator-regulator problem imposes design constraints on the attitude control system, in the sense that the mission will end prematurely if the attitude and gyro drift rate diverge too rapidly. This is not really the case, however. The apparent instability arises only because Eqs. (3) and (7) are extremely simple representations of spacecraft and gyro dynamics. While perhaps not unreasonable over a short time interval, this model is unrealistic over long periods of time. A simple modification of the model involves addition of the term $-d/\tau$ to the right-hand side of Eq. (3). The consequent modeling of gyro drift as a first-order Gauss-Markov process with correlation time τ, rather than as a random walk, eliminates the instability problem and typically has a negligible effect on the optimum estimator gains, for large τ. The key point is that a satisfactory steady-state estimator can be derived from a model which is simple and is reasonably accurate only over short periods of time.

Acknowledgments

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Reply by Author to L.J. Wood

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THE author is thankful to Dr. Wood for his valued comments. It is rightly pointed out that the instability of the state covariance is due to the particular choice of the system model. The model is approximate but has the advantage of easy on-board implementation. Since attitude covariance convergence is not assured directly, this system model fails to bring out the long-term performance of the attitude determination scheme. The use of so many equations to show the state covariance divergence is to find quantitatively the amount of uncertainty expected of such a system model at any given instant of time.

Even if a first-order Gauss-Markov process is used instead of a white noise to represent the random change in the gyro bias drift rate, the divergence of the attitude covariance cannot be got rid of, since the system matrix pair (A,B) is still neither controllable nor stabilizable with drift rate feedback control alone.

In order to ensure covergence of the attitude covariance, the response of the attitude control system may be modelled with a first-order lag, besides the gyro bias drift rate as a first-order Gauss-Markov process.

$$\dot{\psi} = -\psi/\tau_1 + d + u + \eta_v \qquad d = -d/\tau_2 + \eta_u$$

where both τ_1 and τ_2 are large compared to the filter update interval T.

The system is still not controllable, but it is now stabilizable with drift rate feedback alone. The attitude covariance of the combined regulator-estimator now remains bounded as $t \to \infty$. Analytical results on the long-term performance of the algorithm with the modified model can then be derived.

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Comment on "Orbital Decay Due to Drag in an Exponentially Varying Atmosphere"

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THE paper by A.J.M. Chakravarty[1] describes the formal application of the two-variable Asymptotic Expansion Procedure (AEP) to the problem of two-dimensional orbital motion of a ballistic vehicle as perturbed by aerodynamic interaction with a variable-density atmosphere. The purpose of this Technical Comment is to help place the reported study in a somewhat wider context, and thereby draw attention to some interesting and hopefully useful results obtained in earlier studies on the same subject. Specifically, the wider context sought is that of other, earlier, and formal applications of the multivariable AEP to the problem of aerodynamically perturbed satellite motion.

A special case of the multivariable AEP is the two-variable AEP (used in Ref. 1), developed by Kevorkian[2] (see also Refs. 4 and 6). Here, two linear time-like "clocks" are used: a "fast clock" $\tau_1 \triangleq \tau$, and a "slow clock" $\tau_2 \triangleq \epsilon\tau$ $(0 \le \epsilon \ll 1)$, where τ represents the independent variable. In studies of two-dimensional, aerodynamically perturbed satellite motion the independent variable typically represents the central angle (between a suitable in-plane inertial reference vector and the radius vector), whereas the small parameter ϵ is usually defined as the ratio of drag to weight at initial time.

Kevorkian applied the two-variable AEP to the problem of two-dimensional, aerodynamically perturbed motion of a ballistic satellite in a constant-density atmosphere[2] (see also Ref. 3, p. 3). Kevorkian's problem formulation was then generalized by Simmons,[3] who included the effects of aerodynamic lift on the satellite orbit (see also Ref. 4, pp. 264-

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†Where "C" indicates Chakravarty's equations in Ref. 1.

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269). In both cases the basic equations are of the same type as, though not identical to, those of Chakravarty [see Eqs. (9c) and (10c)],† and terms of the second order and higher in the eccentricity were deleted. Both zeroth- and first-order terms of the solution were obtained. The solution was completed in Appendix A of Ref. 5.

A more general multivariable AEP, involving an a priori undefined number of clocks which are in general nonlinear, was developed by Van Woerkom and applied to Simmons' problem of lifting satellite motion in a constant-density atmosphere.[5,6] One of the basic equations used was of the type of Eq. (9c); the other basic equation described the evolution of the square of the nondimensionalized inverse orbital angular momentum, $K(\theta)$. Again, higher-order terms in the eccentricity were deleted, and zeroth- and first-order terms of the solution were obtained.

It is worth noting that use of $U(\theta)$ and $K(\theta)$ as dependent variables reduces algebraic labor considerably. Indeed, with the definitions (notation as in Ref. 1)

$$U \triangleq R_0/R \qquad K(\theta) \triangleq GMR_0(R^{-2}\mathrm{d}T/\mathrm{d}\theta)^2 \tag{1}$$

one obtains, in place of Eqs. (9c) and (10c),

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u=K \tag{2}$$

$$\frac{\mathrm{d}K}{\mathrm{d}\theta}=2\epsilon\frac{K}{U}\left\{1+\left(\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}\theta}\right)^2\right\}^{1/2}\exp\left(\frac{U-1}{H_sU}\right) \tag{3}$$

Following Refs. 5 and 6, one finds that application of the two- (or multi-)variable AEP results in a system of zeroth- and first-order partial differential equations that exhibits considerably more simplicity than the equivalent Eqs. (9c) and (10c). If one then evaluates the effort needed to integrate Eq. (19c), then the simplification obtained by using $K(\theta)$ rather than $t(\theta)$ as dependent variable will be appreciated even more.

Subsequent to Ref. 5, the same general multivariable AEP with an a priori undefined number of (possibly nonlinear) clocks was applied by Van Woerkom[6] to a generalization of the previous study; namely, one now including an semi-exponentially varying atmospheric density model. Again, lifting satellites were considered.

The density model is that of Bruno, who studied the secular evolution of orbital parameters due to aerodynamic perturbations. It is of the form[7]

$$\rho=\rho_0U\exp\left(\frac{U-1}{H_s}\right) \tag{4}$$

Since eccentricity is small in all studies mentioned, one can readily prove asymptotic equivalence (in the eccentricity) between this density model and the purely exponential one used in Ref. 1. In any case, the scale height H_s is to be determined by some method of judiciously matching the density model to the supposedly accurately known true density profile in the perigee region.

One area usually (and in those cases unfortunately) not dwelt upon in publications on the application of generalized perturbation methods (such as the poweful two- and multivariable AEP's cited above) is that of determination of the actual time-like region of uniform asymptotic validity of the solution obtained for practical, nonvanishing numerical values of the small parameter ϵ (such as $\epsilon=1.4\times10^{-6}$ in Ref. 1, or $\epsilon=10^{-4}$ in Refs. 5 and 6).

Finally, some remarks of a more detailed nature:

1) The value of 0.04 for C_D used in the numerical example seems unrealistically low. Rather, one would expect to see much higher values, typically $C_D=2$ to 3 (see Ref. 8).

2) The solution presented in Ref. 1 appears to violate the initial condition $\mathrm{d}U/\mathrm{d}\theta=0$ at $\theta=0$. Indeed, differentiation of Eq. (20a-c) and subsequent substitution of Eqs. (23-25c) gives

$$\frac{\mathrm{d}U}{\mathrm{d}\theta}(0)=\epsilon\{2+e(0)\}\exp(\beta)>0 \tag{5}$$

3) Use of the eccentricity e and the argument of the perigee ω as slowly varying functions, Eqs. (20c), is not always recommendable, since ω becomes undetermined as e goes to zero (which does occur in the case of aerodynamic perturbations). This property can also be read off immediately from the structure of Gauss' form of the Lagrangian planetary equations.[9] When dealing with small eccentricities[1-6] one might prefer the use of a formulation of the type[6]

$$U_0(\theta,\tilde{\theta})=p^2\{1+g_1\sin\theta+g_2\cos\theta\} \tag{6}$$

where p, g_1 and g_2 turn out to depend on $\tilde{\theta}$. Note the relationships $g_1\equiv e\cdot\sin\omega$ and $g_2\equiv e\cos\omega$.

4) Consider the density relation, Eq. (6c), as used in Eq. (19c). Substitution of Eq. (20a-c) gives

$$H_s\ln(\rho/\rho_0)=1-p^{-2}\{1+e\cos(\theta-\omega)\}^{-1} \tag{7}$$

where p, e, ω are functions of $\tilde{\theta}$ only.

An order-of-magnitude analysis is now carried out. From Eq. (20a-c) one has $p^{-2}=1+e$, at least initially. Upon substitution of this relation in Eq. (7) one obtains, to the first order in the eccentricity,

$$H_s\ln(\rho/\rho_0)=e\{\cos(\theta-\omega)-1\}+\mathcal{O}(e^2) \tag{8}$$

On the other hand, if one integrates Eq. (19c) one finds that the following approximation has been made:

$$H_s\ln(\rho/\rho_0)=H_s\beta=1-p^{-2}=-e \tag{9}$$

where the at least initially valid relation $p^{-2}=1+e$ has been introduced.

There exists a clear discrepancy between Eqs. (8) and (9). In the latter, the dependence of ρ on the fast clock θ is completely absent, for no apparent reason. Thus, use of Eq. (9) in Ref. 1 amounts to the use of a quasi-constant density model of the functional type $\rho(\tilde{\theta})$, rather than $\rho(\theta,\tilde{\theta})$. The presence of the "fast" cosine term in Eq. (8) is of great importance, however. It shows that density does peak near perigee, and that the angular distribution of the density is periodic but not sinusoidal at all.

Integration of the correctly expanded density relation will be found to lead to the appearance of Modified Bessel Functions of the first kind.[6]

Remark 2 above prompts another, more general remark on the importance of proper "bookkeeping" of the order of magnitude ϵ^n ($n=0,1,2,\ldots$) of the various terms involved. To be systematic, one should obtain the analytic solution for $U_1(\theta,\tilde{\theta})$ (e.g.) as well. The dependence of the "constants" in U_1 on the slow clock $\tilde{\theta}$ is at that stage still unknown. One then applies the concept of "Restriction of Number of Clocks" introduced in Refs. 5 and 6, to obtain the full first-order asymptotic solution

$$U(\theta)=U_0(\theta)+\epsilon U_1(\theta)+\mathcal{O}(\epsilon^2) \tag{10}$$

Details about these and other aspects of multivariable Asymptotic Expansion Procedures, comments on the historical development of these AEP's since Euler (1772), applications to gravitationally as well as aerodynamically perturbed satellite trajectories, and an extensive and critical discussion of relevant publications, can be found in Ref. 6.

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Reply by Author to P.Th. L.M. van Woerkom

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THE author wishes to thank Paul van Woerkom for his interest in our work and for his comments. At the time the original manuscript for our paper[1] was prepared, van Woerkom's work, which appeared in the European literature, was unknown to us and we apologize for any confusion that may have resulted from not citing his work in our references.

The use of $K(\theta)$ rather than $t(\theta)$ as a dependent variable is a matter of choice, and in obtaining the solution to order unity (which was our goal), we did not confront any major problem in dealing with the algebra. In addition, we are not able to justify the use of the proposed density model which is referred to in an obscure Princeton University seminar. The assumption of a standard atmospheric model leads to our Eq. (6).

In his Comment, van Woerkom notes that $u'(0)\neq 0$ but is of order ϵ. We were interested in obtaining the solution only to order unity, however, and $u'(0)=0$ is definitely correct to that order.[2] We are grateful to van Woerkom for pointing out that ω becomes undetermined as e goes to zero. But, since the eccentricity is away from zero (though very small) in our analysis, the use of e and ω or g_1 and g_2 (van Woerkom's notation) is again a matter of choice.

Finally, in integrating Eq. (19) of Ref. 1, we have made use of the fact that $1+e\cos(\theta-\omega)$ is approximately equal to unity in magnitude (since $e(0)$ is small and e is monotonically decreasing) and therefore β is independent of θ. But we acknowledge the fact, as aptly pointed out by van Woerkom, that a very pertinent piece of information, namely that angular distribution of the density is periodic, is lost. Nonetheless, the results shown in Fig. 1 of Ref. 1 are not much affected.

In conclusion, it is true that to determine $t(\theta)$ to order ϵ^n, we must know v_{n+1} completely, and this requires examining the equations for u_{n+2} and v_{n+2}.

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Errata

Semianalytic Theory for a Close-Earth Artificial Satellite

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THE last three equations of Eqs. (2) should read

$$(\dot{\Omega})_D = -\frac{1}{2}B\rho V\frac{r^2\omega_a}{\sqrt{\mu a(1-e^2)}}\sin u\cos u$$

$$(\dot{\omega})_D = -\frac{1}{2}B\rho V\frac{\sin f}{e}\left[2-\frac{r^2\omega_a\cos i}{\sqrt{\mu a(1-e^2)}}\left(2+e\cos f+\frac{e\sin u\ \cos u}{\sin f}\right)\right]$$

$$(\dot{M})_D = \frac{1}{2}B\rho V\left[\cdots\right]$$

The first-order short-periodic variations for mean anomaly given in Appendix A should read

$$M_{sp} = -\frac{3}{2}J_2\left(\frac{R}{p}\right)^2\frac{\sqrt{1-e^2}}{e}\left\{\cdots\right\}+\cdots$$

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